

Towards the Hall viscosity of the Fermi-liquid-like phase at the filling factor of $1/2$

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We discuss the Berry curvature calculations of the Hall viscosity for the unprojected to the lowest Landau level wave function of the Fermi-liquid-like state. We conclude, within assumptions made, that in the linear response, with small deformation of the system and in the thermodynamic limit, the Hall viscosity takes the value characteristic for the Laughlin states. We present arguments that the value is the same even for general deformations in the same limit.

I. INTRODUCTION

The Hall viscosity may represent an additional invariant by which we can characterize quantum Hall states. In the work of Read¹ it follows directly from conformal field theory (CFT) description of quantum Hall states and it is related to the conformal spin of the CFT field that is associated with the electron description. In this way some non-unitary CFTs and corresponding, presumably gapless, quantum Hall states may have a well-defined Hall viscosity. The question is whether the “quantization” of the Hall viscosity will persist as we modify model Hamiltonians for which the states are exact zero energy states i.e. whether these phases despite being gapless have an invariant - the Hall viscosity. We still do not have a definite answer for gapful cases, but this question seems related and relevant. The second question is: if a quantum Hall state cannot be expressed by a CFT correlator like the Fermi-liquid-like state at $1/2$, whether it may still have the invariant - Hall viscosity equal to the value of Laughlin states which simple considerations of orbital spin of electron may motivate¹. On the other hand in that case we may think that the Hall viscosity is the limiting value of the series of Hall viscosities of the states of Jain series that leads to the Fermi-liquid-like state. If the Hall viscosity is proportional¹ to the shift of the state multiplied by its filling factor the limiting value will diverge and certainly we will not have a finite (“quantized”) value associated with other states constructed as CFT correlators.

In this paper we will discuss the Hall viscosity of the Fermi-liquid-like state² and phase³ at $\nu = 1/2$. We will consider the unprojected to the lowest Landau level (LLL) wave function for the ground state introduced in Ref. 2, but at $\nu = 1$ i.e. the Fermi-liquid-like state of bosons⁴. The bosonic and fermionic state share the same underlying Fermi-liquid-like physics. The bosonic version will facilitate the calculations of the Berry curvature and its coefficient, the Hall viscosity, in the case of these gapless systems. In the introductory Section II we will review the basic ansatz for calculating the Hall viscosity in the case of quantum Hall states that was used in Ref. 5. In Section III the Hall viscosity of free Fermi gas is discussed as a step towards the calculation for the Fermi-liquid-like state. In Section IV the Hall viscosity as a response to a small deformation of the Fermi-liquid-like state is discussed (a) when the system is quasi-one-dimensional, (b) in the case when the Fermi surface is rectangular, and finally (c) in the case of interest i.e. when the Fermi surface is isotropic and circular. The next section, Section V, discusses the Hall viscosity of a system under a general deformation, and Section VI the importance of the inversion symmetry for the neutral part as an effective symmetry for composite fermions (CFs) that is present in the systems with rectangular shape, which are deformed. Section VII contains a discussion of results and conclusions.

II. HALL VISCOSITY OF QUANTUM HALL STATES

The approach^{1,6-8} to the Hall viscosity that we find in the literature relies on calculating the Berry curvature of shear deformations of the ground state that is adiabatically transformed. This means an assumption is made that the state is non-degenerate along the process. This can be assured if the system has a gap which is the characteristic of usual quantum Hall states. The shear deformations are examined by following how the quantum liquid is spread out in the deformed geometry of a torus (we will discuss the boundary conditions (BCs) shortly)- see Figure 1.

The parameters V and $\tau = \tau_1 + i\tau_2$ describe the deformation from the reference point $V = 1$ and $\tau = i$. There are two equivalent approaches in the literature. The first one is (a) to stay in the coordinate space that we begin, with “old” coordinates $(x, y) \in [0, 1] \times [0, 1]$, but study the solutions of a deformed Hamiltonian. This Hamiltonian is the usual local Hamiltonian in “new” coordinates (x', y') but now expressed in terms of the old coordinates using the coordinate transformation $x' + iy' = (\frac{V}{\tau_2})^{1/2}(x + \tau y)$, associated with the deformation. The usual (periodic) boundary conditions are applied. The second approach is (b) to apply the deformed boundary conditions i.e. a new condition⁹ in the direction of τ and stay with the same local (undeformed) Hamiltonian. Both approaches lead to the same

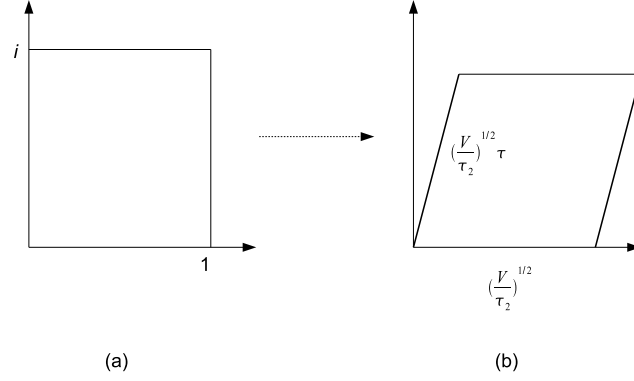


FIG. 1: Deformed torus

deformed ground state Ψ which should be used in the formula for Hall viscosity with the Berry curvature:

$$\eta^A = 2 \operatorname{Im} \langle \partial_{\tau_1} \Psi | \partial_{\tau_2} \Psi \rangle, \quad (1)$$

calculated at the reference point $V = 1$ and $\tau = i$. To recover the physical units we should multiply with $\frac{\hbar}{L_x L_y}$ where L_x and L_y are in a general case the lengths associated with a rectangular system. In the following we will study such a general geometry in which the deformations from a rectangle with lengths L_x and L_y are made - see Figure 2.

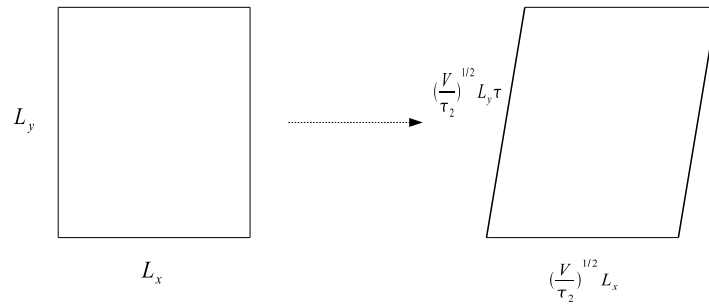


FIG. 2: Deformed rectangle

If we consider a gapped system of non-interacting electrons that fill the LLL to get the Hall viscosity we have to sum the contributions from each single particle state in the LLL. The wave functions that describe the way how a

single particle wave function is changed as the geometry of the finite-volume system is varied (Figure 2) are

$$\Psi_j = \sum_{k=-\infty}^{+\infty} \exp\{i \frac{(X_j + kL_y)x}{l_B^2} + i \tau \frac{(X_j + kL_y - y)^2}{2l_B^2}\}, \quad X_j = \frac{2\pi l_B^2 j}{L_x}. \quad (2)$$

We did not include the normalization of each wave function that is labeled by an integer $j = 0, \dots, N_s - 1$, where $N_s = \frac{L_x L_y}{2\pi l_B^2}$ i.e. the number of flux quanta through the system; in our case this number is equal to the number of electrons. The coordinates x and y are old coordinates and the wave functions satisfy the ordinary periodic BC (PBC) in x -direction:

$$\Psi_j(x + L_x) = \Psi_j(x), \quad (3)$$

and the magnetic BC in y - direction:

$$\Psi_j(y + L_y) = \exp\{i \frac{L_y x}{l_B^2}\} \Psi_j(y). \quad (4)$$

We will relax⁵ the demand for the magnetic BC (or we may take large L_y limit¹⁰ in Eq.(2)). The wave function becomes simpler in this cylinder geometry:

$$\Psi_j = \exp\{i \frac{X_j x}{l_B^2} + i \tau \frac{(X_j - y)^2}{2l_B^2}\} \times \frac{(\tau_2)^{1/4}}{(l_B \sqrt{\pi})^{1/2}}, \quad (5)$$

where we included the normalization. We have a set of orthonormal wave functions which can be used for the calculation of the Berry curvature as a sum of contributions of each single particle state. As we prove⁸ in the Appendix A, if the wave function is non-analytic i.e. non-holomorphic in τ variable only in its normalization i.e. can be expressed as $\Psi = \frac{1}{\sqrt{Z}} f(\tau, x, y)$, its contribution to the Hall viscosity is

$$\frac{\hbar}{L_x L_y} \times \frac{1}{2} \left(\frac{\partial^2}{\partial \tau_1^2} + \frac{\partial^2}{\partial \tau_2^2} \right) \ln Z. \quad (6)$$

Specifying to our set, Eq.(5), the sum of all contributions is

$$\eta^A = \frac{\hbar n}{4}. \quad (7)$$

Therefore we recovered the well-known result (for $\nu = 1$ QHE) using the cylinder geometry and it will be the same even if we were applying the so-called “thin-torus limit” i.e. cylinder limit for which $L_x \rightarrow 0$.

III. HALL VISCOSITY OF FREE FERMI GAS

Classically and in the adiabatic response theory it is expected⁶ that the system with time-reversal symmetry does not have the asymmetric (Hall) viscosity. We study the asymmetric viscosity of the free Fermi gas in the following. We will use the Berry curvature formula to calculate the Hall viscosity even for this system assuming that in the adiabatic response, when we probe a finite fraction i.e. small finite system, the ground state stays non-degenerate as τ is varied at least for a small interval before a reconfiguration of the Fermi surface. In the case of free Fermi gas the tiny gap $\Delta \sim \frac{1}{L^2}$, where L is the length of the system, $L = \max\{L_x, L_y\}$ keeps the filling of the Fermi sea intact at least for values of V, τ_1 , and τ_2 in the neighborhood of $V = 1$ and $\tau = i$. Therefore because the Hall viscosity is the Berry curvature at a specific point in the parameter space and not an integral of it in the same space the demand for the ground state being non-degenerate can be relaxed to the same requirement in the neighborhood of the unperturbed point. In fact the linear response theory leads to the Berry curvature formula⁵.

In our case of the free Fermi gas we need a small enough system. We will adopt the approach in Ref. 6 and study the deformed Hamiltonian:

$$H = -\frac{1}{V\tau_2} [|\tau|^2 \partial_x^2 - 2\partial_x \partial_y + \partial_y^2], \quad (8)$$

on space $(x, y) \in [0, L_x] \times [0, L_y]$, with periodic boundary conditions. We seek the solutions in the form:

$$\exp\{ik_x x' + ik_y y'\} = \exp\{i(k_x x + k_x \tau_1 y + k_y \tau_2 y)\} \sqrt{\frac{V}{\tau_2}}, \quad (9)$$

as we demand that locally we have the same equation irrespective whether we work with “old” or “new” coordinates. The eigenvalues are $\epsilon(\mathbf{k}) \sim k_x^2 + k_y^2$. But the demand for PBCs and orthogonality leads to

$$k_x = \frac{2\pi}{L_x} \sqrt{\frac{\tau_2}{V}} m \quad \text{and} \quad k_y = 2\pi \frac{1}{\sqrt{V\tau_2}} \left(\frac{n}{L_y} - \frac{m}{L_x} \tau_1 \right), \quad (10)$$

where m and n are integers. Therefore the quantized energy levels are:

$$\epsilon(\mathbf{k}) = \epsilon(n, m) = \frac{(2\pi)^2}{V\tau_2} \left[\tau_2^2 \frac{m^2}{L_x^2} + \left(\frac{n}{L_y} - \frac{m}{L_x} \tau_1 \right)^2 \right]. \quad (11)$$

Though the deformation τ modifies the eigenvalues, eigenstates are independent of it, and this leads to zero value for Berry curvature and the Hall viscosity as expected in a time-reversal invariant system. As we recovered the result that is valid for a system of any size, we will use the obtained description and formulas even in the large N limit in the following.

IV. HALL VISCOSITY OF FERMI-LIQUID-LIKE STATE

We will consider the bosonic Fermi-liquid-like state at the filling factor $\nu = 1$ because we expect that the conclusions will not depend on the kind of the Laughlin-Jastrow factor in the wave function. Therefore we study wave function:

$$\Psi = \Psi_L^{\nu=1} \times \text{Det}(\exp\{i\vec{k}_i \vec{r}_j\}), \quad (12)$$

which is not normalized, and $\Psi_L^{\nu=1}$ is the Vandermonde determinant and $\text{Det}(\exp\{i\vec{k}_i \vec{r}_j\})$ the Slater determinant of free waves. Therefore we study the unprojected to LLL wave function. We will assume the following evolution of the wave function under deformation τ : Each factor will evolve according the deformed single-particle Hamiltonians: (a) the one with magnetic field as in Avron et al. with magnetic boundary conditions in the case of the evolution of the part that “sees” the magnetic field i.e. Vandermonde determinant and (b) the Hamiltonian given in Eq.(8) with PBCs that governs the evolution of the part with plane waves. Therefore we assume separate evolutions that we know very well. As we study the small deformations of a rectangular system and plane waves do not depend on it, the most important question is what is the shape of the Fermi sea of the unperturbed finite Fermi system in a rectangular geometry. This is a difficult question though we believe that in the thermodynamic limit the Fermi sea will assume its circular, isotropic shape. But if we stay with the shape of a rectangle even in the thermodynamic limit, as we should do just as in the Laughlin case when we apply the Berry curvature formula⁵ the question is still there¹³. It has to be resolved only by studying the full interacting system in the LLL. Here we will be studying (A) a limiting case of thin torus (cylinder), (B) a system with rectangular shape of its Fermi surface and then reach conclusions in the case (C) which is an isotropic, circular Fermi surface.

Before that we will analyze the Berry curvature formula for the wave function in Eq.(12) with the assumed evolution in general terms. Because the part with Slater determinant of free waves does not depend on τ , the expression for the Hall viscosity is again

$$\eta_A = \frac{\hbar}{L_x L_y} \frac{1}{2} \Delta \ln Z, \quad (13)$$

where Z is the norm of the wave function in Eq.(12) and the derivatives are calculated at $V = 1$ and $\tau = i$.

A. Thin cylinder limit

The system can be viewed as in Fig. 3

Then because of the PBC in x -direction we have the quantization of the momentum as before

$$X_j = \frac{2\pi l_B^2}{L_x} j, \quad (14)$$

$j = 0, \dots, N_s - 1$ where $N_s = \frac{L_x L_y}{2\pi l_B^2}$. Now we take $L_x \rightarrow 0$ limit along $L_y \rightarrow \infty$ to keep N_s constant. For the Fermi-liquid-like state that means that the neutral fermions in the k -space will form a line along y direction with

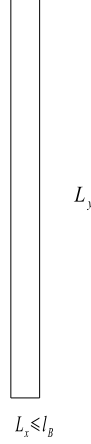


FIG. 3: Thin rectangle limit

two Fermi-points instead of a circle (line) for a Fermi surface. In real space that is described by the following wave function,

$$\prod_{i < j} \sin\left\{\frac{\pi}{L_y}(y_i - y_j)\right\}, \quad (15)$$

where we assumed an odd number of electrons. Notice that there is no x -dependence. Therefore when we ask for the norm of the complete wave function (with the Laughlin-Jastrow factor at $\nu = 1$ -Vandermonde determinant) we get

$$Z = \prod_{i=1}^N \int dy_i \sum_{\sigma \in S_N} \exp\{-\tau_2(y_i - k_{\sigma(i)})^2\} \prod_{k < l} \sin^2\left\{\frac{\pi}{L_y}(y_k - y_l)\right\}. \quad (16)$$

Under translations of y -variables:

$$Z = \prod_{i=1}^N \int dy_i \exp\{-\tau_2 y_i^2\} \sum_{\sigma \in S_N} \prod_{k < l} \sin^2\left\{\frac{\pi}{L_y}(y_k - y_l + k_{\sigma(k)} - k_{\sigma(l)})\right\}. \quad (17)$$

Due to the Gaussian factors in y -integration for $\tau_2 \sim 1$ we can assume that relevant values of y 's in the product are $y_i \lesssim l_B, \forall i \in [1, N]$. Because $|k_{\sigma(k)} - k_{\sigma(l)}| \geq \frac{2\pi l_B^2}{L_x}$, when $L_x \rightarrow 0$ we can neglect the presence of y 's in the sine functions and due to the scalings, $\tau_2 y_i \rightarrow y_i$, we recover the result for the Hall viscosity identical to the integer quantum Hall effect at $\nu = 1$.

B. The system with a rectangular shape of its Fermi surface

The Fermi gas with a rectangular shape of its Fermi surface may be rather artificial but as we already discussed (a) this shape may appear in small systems with rectangular boundaries and (b) the conclusions reached and constructions applied to this system will serve as a stage for discussing the problem with rectangular boundaries in the thermodynamic limit and circular shape of the Fermi surface.

Let us assume that we have a Fermi surface of a rectangular shape where, for simplicity, we take that the length and width are the same and proportional to $\sqrt{N} \in \mathbb{Z}$. (To retain PBCs (instead of antiperiodic BCs) we may demand that \sqrt{N} is an odd number.) The ground state function of the ideal gas has to be an eigenvector under inversion symmetry: $y_i \rightarrow -y_i$ and $x_i \rightarrow x_i, \forall i$ (or $y_i \rightarrow y_i$ and $x_i \rightarrow -x_i, \forall i$) and that constrains its form to two possibilities:

(1)

$$\mathcal{A} \left\{ \prod_{\text{over slices in } k \text{ space}} \left[\prod_{i < j; i, j \in \text{slice}} \sin\left\{\frac{\pi}{L_y}(y_i - y_j)\right\} \cos\left\{\frac{\pi}{L_x}(x_i - x_j)\right\} \right] \right\}, \quad (18)$$

where slices are lines in k - space, along k_x and k_y direction, of length \sqrt{N} each and to each one is assigned \sqrt{N} number of particles, see Figure 4. For a fixed \sqrt{N} number of particles we have two slices or lines symmetrically

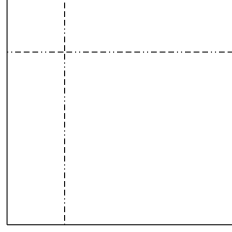


FIG. 4: Rectangular Fermi surface and two slices with the same group of particles

positioned in k space around $k_x = -k_y$ line (Figure 4). So as a first step we divide particles in \sqrt{N} slices (lines, groups), and at the end we antisymmetrize (\mathcal{A}) that construction in the curly brackets, which represents a particular division into \sqrt{N} groups. See an example with 4 particles in Appendix B. (We introduced slicing in k space although at this point it seems redundant - only division in \sqrt{N} groups and later antisymmetrization is all that is in Eq.(18); slicing in k - space is helpful to introduce and analyze more general Fermi surfaces as we will see later on.)

The second possibility (2) is with x 's and y 's interchanged. If the width and length are not the same, for example $L_y > L_x$ then for a single slice of $\sqrt{N} \frac{L_y}{L_x}$ integers (integers denote particles of the particular slice or group), S_y , we have to symmetrize, in addition, smaller slice of $S_x \subset S_y$ integers, with $\sqrt{N} \frac{L_x}{L_y}$ of them, i.e.

$$\prod_{i < j; i, j \in S_y} \sin\left\{\frac{\pi}{L_y}(y_i - y_j)\right\} S \left\{ \prod_{k < l; k, l \in S_x} \cos\left\{\frac{\pi}{L_x}(x_k - x_l)\right\} \right\}, \quad (19)$$

so that x - part is also symmetric under permutations inside S_y .

Then our norm, i.e. Z , for the compressible quantum Hall state at $\nu = 1$ becomes a sum of terms, each representing two fixed permutations σ, σ' of N integers as in the following,

$$\prod_{i=1}^N \int dy_i \prod_{l=1}^N \exp\left\{-\frac{\tau_2}{2}(y_l - k_{\sigma(l)})^2\right\} \prod_{p=1}^N \exp\left\{-\frac{\tau_2}{2}(y_p - k_{\sigma'(p)})^2\right\} \times \\ \prod_{\text{over slices}} \left\{ \prod_{k < l; k, l \in \text{slice}} \sin\left\{\frac{\pi}{L_y}(y_k - y_l)\right\} \right\} \prod'_{\text{over slices}} \left\{ \prod_{p < q; p, q \in \text{slice}} \sin\left\{\frac{\pi}{L_y}(y_p - y_q)\right\} \right\} \quad (20)$$

where we suppressed (did not write) the part that corresponds to x - integration. (Note now $\sigma \neq \sigma'$ in general due to the x - dependence of the wave function describing Fermi sea.) Now we can shift each y_i , by $(k_{\sigma(i)} + k_{\sigma'(i)})/2 \equiv \bar{k}_i$ and estimate how the scaling with τ_2 can be affected with the Fermi sea part. The factors that come out, $\exp\{-\frac{\tau_2}{4}(k_{\sigma(i)} - k_{\sigma'(i)})^2\}$, (besides the Gaussians in y 's) will suppress the contributions of terms, Eq.(20), for which σ and σ' differ too much and in the following we will assume $|\sigma(i) - \sigma'(i)| \ll \frac{L_x}{l_B}$. With this in mind we concentrate on a single slice:

$$\prod_{k < l; k, l \in \text{slice}} \sin\left\{\frac{\pi}{L_y}(y_k - y_l + \bar{k}_k - \bar{k}_l)\right\}. \quad (21)$$

Now we ask again the question when $|\bar{k}_k - \bar{k}_l| \lesssim l_B$. Because in this case L_x is not small we can have $\bar{k}_i \lesssim l_B$ for $\sigma(i) \lesssim \frac{L_x}{l_B}$ and an estimate can be that this can happen for all pairs $\sigma(l)$ and $\sigma(k)$ in Eq.(21) for which $\sigma(l), \sigma(k) \lesssim \frac{L_x}{l_B}$ and we might think that there are $N_p \approx \frac{L_x}{l_B}(\frac{L_x}{l_B} - 1)/2$ of them. But this is an overestimate for the construction in Eq.(18) because by making division in slices we do not have the factor $\sin\{\frac{\pi}{L_y}(y_k - y_l)\}$ for each pair of particles. As we do not have as many pairs as $N(N-1)/2$, but because of slicing only around $\sim L_x \times L_y^2/l_B^3$ (or $N^{3/2}\frac{L_y}{L_x}$), the N_p should be reduced¹¹ by L_x and therefore is not of the order of N , which would pose a problem in the scaling argument for the Hall viscosity and influence its final value. Therefore we argued that we can model the contribution of each term as in Eq.(20) with $\sigma \approx \sigma'$ as

$$\sim \frac{1}{\tau_2^{N/2+\alpha}} I(L_x, L_y), \quad (22)$$

where $\alpha \lesssim \frac{L_x}{l_B}$. Even if the specific value of α depends on the choice of grouping of particles for slicing in Eq.(18), by extracting a leading contribution in N we can recover the same result for the Hall viscosity as before. We in fact are taking the large N limit before the limit $\tau \rightarrow i$, which is allowed¹² but implies the same value of the Hall viscosity only in this limit. More precisely, if we do not assume the effective reduction of each $\sin\{\frac{\pi}{L_y}(y_k - y_l + \bar{k}_k - \bar{k}_l)\}$ to either $\sin\{\frac{\pi}{L_y}(y_k - y_l)\} \sim \frac{\pi}{L_y}(y_k - y_l)$ or $\sin\{\frac{\pi}{L_y}(\bar{k}_k - \bar{k}_l)\}$, in the end the term in Eq.(20) can be expressed as a series with each member of the form as in Eq.(22), where again $\alpha \lesssim \frac{L_x}{l_B}$, and the argument follows. This is possible because for any $|\bar{k}_k - \bar{k}_l| \lesssim l_B$ and, as we have due to the shifts and Gaussians $|y_k - y_l| \lesssim l_B$, we can approximate

$$\sin\{\frac{\pi}{L_y}(y_k - y_l + \bar{k}_k - \bar{k}_l)\} \approx \frac{\pi}{L_y}(y_k - y_l + \bar{k}_k - \bar{k}_l), \quad (23)$$

and an expansion in the differences in y 's i.e. $\frac{\pi}{L_y}(y_k - y_l)$ follows.

C. The system with circular Fermi surface

Our expectation is that the composite fermions will make an isotropic, circular Fermi surface even in the thermodynamic limit of the system with rectangular boundaries. Nevertheless in this case we have to demand that the ground state of the system retains the inversion symmetry of the system in its neutral sector, i.e. that the Fermi part of the ground state wave function is an eigenvector under $y_i \rightarrow -y_i$ and $x_i \rightarrow x_i, \forall i$, transformation. The rectangular shape is a feature of the system on which the shear transformation is applied. Such a ground state wave function can be constructed by a generalization of the construction in the previous case (B) given in Eq.(18). Now the two slices along k_x and k_y direction are as in Figure 5. A group of particles, its number equal to the length of the two slices is assigned to them. As we sweep the whole circle with these two slices we make a certain division of all particles into groups that correspond to slices and therefore, to make the wave function antisymmetric under particle exchange, we need an overall antisymmetrization as in Eq.(18). The same arguments as in the case with rectangular Fermi surface can be applied here and lead to the conclusion that the Hall viscosity of the Fermi-liquid-like quantum Hall state is unaffected by the presence of the Fermi sea in the ground state function. Namely all estimates that we did in the rectangular case will be modified by geometrical factors that will not affect the conclusion on the leading behavior in the thermodynamic limit. To illustrate what we mean by geometrical factors let us consider a Fermi surface that is a square with \sqrt{N} length of each side. In that case the number of pairs is $\sqrt{N} \times \sqrt{N}(\sqrt{N} - 1)/2$, but in the case of a circle, which delineates the same volume equal to N , we will have for the same quantity $\frac{16}{\pi^{3/2}}N^{3/2} + 4N$ i.e. the same leading behavior $\sim N^{3/2}$ up to a numerical - geometrical factor.

V. THE HALL VISCOSITY AT $\tau \neq i$

The most important question when considering the question of the Hall viscosity for the Fermi-liquid-like state is whether it is dependent on τ or maybe it is independent of the geometry (τ), which is a remarkable property of the integer quantum Hall state⁶ at $\nu = 1$ and other quantum Hall states¹ that exhibit Hall conductance plateaus.

When considering arbitrary τ we have to start with the deformed Fermi surface as it follows from the deformed dispersion relation in Eq.(11). To simplify the notation we will take $L_x = L_y = L$ or, in general, that m and n carry factors connected with the lengths and may not be integers. Therefore we write the dispersion relation $\epsilon(\tau)$ as

$$\epsilon(\tau) = \frac{(2\pi)^2}{VL^2\tau_2}[\tau_2^2 m^2 + (n - \tau_1 m)^2], \quad (24)$$

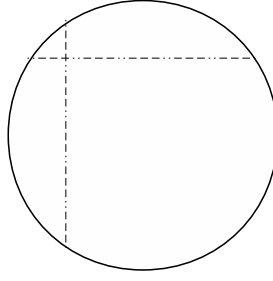


FIG. 5: Circular Fermi surface and two slices with the same group of particles

or if we absorb the scaling factor $f(\tau) = \frac{(2\pi)^2}{VL^2\tau_2}$ and define $e(\tau)$ as $\epsilon(\tau) = f(\tau)e(\tau)$ we may focus on the dispersion relation expressed as

$$e(\tau) = \tau_2^2 m^2 + (n - \tau_1 m)^2. \quad (25)$$

The equation $\epsilon_F = e_F f$ where ϵ_F is the fermi energy defines the (deformed and scaled) Fermi surface i.e.

$$e_F \equiv e = \tau_2^2 m^2 + (n - \tau_1 m)^2, \quad (26)$$

which is illustrated in Figure 6.

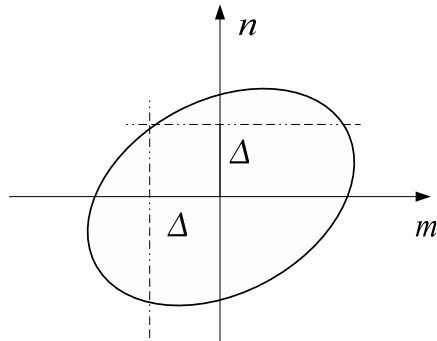


FIG. 6: Deformed Fermi surface and two slices with the same group of particles

We find that the maximum values of m and n (that belong to points on the Fermi surface) are $m_{max} = \frac{\sqrt{e}}{\tau_2}$ and $n_{max} = \frac{\sqrt{e}}{\tau_2}|\tau|$, respectively. And for $m = -\Delta$ we have corresponding $n = -\tau_1\Delta \pm \sqrt{e - \tau_2^2\Delta^2}$ and for $n = \Delta$ we have $m = (\tau_1\Delta \pm \sqrt{e - \tau_2^2\Delta^2})/(|\tau|^2)$. This implies that if we keep $|\tau| = 1$ we would have the same length for corresponding two slices along k_x and k_y directions that we introduced before. To simplify the discussion in the following we will assume that this is the case i.e. that due to $|\tau| = 1$ we have the symmetry under inversion around the axis defined by $n = \tau_1 m$.

As our deformed Hamiltonian in Eq.(8) has the symmetry under simultaneous transformations $\tau_1 \rightarrow -\tau_1$ and $y \rightarrow -y, x \rightarrow x$ or $\tau_1 \rightarrow -\tau_1$ and $y \rightarrow y, x \rightarrow -x$ our dispersion relation, Eq.(11), has the same symmetry and the corresponding Fermi surface as well. This symmetry has to exist in the ground state, which has to accommodate to the deformed rectangular shape for $\tau \neq i$. For $\tau = i$ the symmetry can be identified as the inversion symmetry around x or y axis that has to be generalized to the case with $\tau \neq i$ for which we need to include also $\tau_1 \rightarrow -\tau_1$ transformation. With this in mind we can come up with a ground state wave function that will have this symmetry in the Fermi part under simultaneous transformations in coordinate and momentum space. Using the slice decomposition that is illustrated in Figure 6 for the deformed Fermi surface and with the simplifying assumption $|\tau| = 1$ the Fermi part will look like

$$\mathcal{A} \left\{ \prod_{\text{over slices in k space}} \left[\prod_{i < j; i, j \in \text{slice}} \exp\{ik_y^c \sum_{i \in \text{slice}} y_i\} \sin\{\frac{\pi}{L_y}(y_i - y_j)\} \exp\{ik_x^c \sum_{i \in \text{slice}} x_i\} \cos\{\frac{\pi}{L_x}(x_i - x_j)\} \right] \right\}, \quad (27)$$

where slices in k_x and k_y direction correspond in the manner of Figure 6; to each slice in k_y corresponds the slice of the same length in k_x corresponding to the same group of particles. The exponentials with k_x^c and k_y^c carry the momentum $\vec{k}^c = (k_x^c, k_y^c)$, which is due to the deformation of the Fermi surface and the absence of the inversion symmetries around k_x and k_y axis. The momentum \vec{k}^c lies along the new symmetry line i.e. $k_y^c = \tau_1 k_x^c$ and represents the momentum of the center of the mass of the particles that belong to the particular slice. \mathcal{A} in Eq.(27) is again the overall antisymmetrization that will bring all possible assignments of particles into slices in the final form of the wave function. The construction when x 's and y 's (k_x^c 's and k_y^c 's) are interchanged is also possible and we will discuss that case later.

The complete deformed wave function for the Fermi-liquid-like state has the Gaussian factors of the form:

$$\exp\left\{-\frac{\tau_2}{2}(y_i - k_{\sigma(i)})^2\right\} \quad (28)$$

that enter the integral for Z . If τ_2 is small the y_i can fluctuate being less localized with the Gaussian. So the relevant interval of y_i values in the integral becomes larger and the sequence of approximations for two particles, beginning with the corresponding term in the product in the integral (Eq.(21)), as

$$\sin\left\{\frac{\pi}{L_y}(y_k - y_l + \bar{k}_k - \bar{k}_l)\right\} \approx \sin\left\{\frac{\pi}{L_y}(y_k - y_l)\right\} \approx \frac{\pi}{L_y}(y_k - y_l), \quad (29)$$

are more likely to be allowed. Each term like this will contribute $1/\sqrt{\tau_2}$ when the scaling $\sqrt{\tau_2}y_i \rightarrow y_i, \forall i$ is applied. If, for small enough τ_2 , we assume that for each pair we can do this approximation, in addition to the overall exponent, which we get by the change of variables in the y -integration, of order N , we will get another contribution of order $N^{3/2}$ that would lead to the divergence of the Berry curvature and therefore for finite τ_2 of the Hall viscosity. This is certainly an overestimate but the possibility of divergence seems lurking. Applying arguments similar to the one in Section IV B we come to an estimate that the number of relevant pairs is around $\sim \frac{L_x}{l_B} \frac{1}{\tau_2}$. Therefore only for strong deformations for which $\tau_2 \sim \frac{1}{\sqrt{N}}$, we may expect the departure of the value for the Hall viscosity from the one of Laughlin states. These arguments can not be precisely quantified but suggest that the Hall viscosity of the Fermi-liquid-like state may depend on the value of τ for very large deformations. But as we do apply the large N limit to recover the Laughlin state value for the Hall viscosity as $\tau \rightarrow i$ and because here relevant τ_2 is of the order of $\frac{1}{\sqrt{N}}$, we can expect the same Hall viscosity value in the same limit for any finite τ_2 . Therefore the feature of the Laughlin states that their Hall viscosity is independent of τ may stand as a reflection of their true topological nature due to the comparison with the Fermi-liquid-like state that can recover the same value only in large N limit.

The precise estimate how the Hall viscosity depends on τ (in the case of the Fermi-liquid-like state) is hard to get also because of the exponentials with \vec{k}^c that carry dependence on y_i . (We have to keep in mind that the scaling $\sqrt{\tau_2}y_i \rightarrow y_i, \forall i$ is a purely mathematical transformation of variables under the Z integral and does not affect k -variables.) In the argument above we assumed $\sum_{i \in S} y_i \approx 0$ for each slice S , which might not be the case.

VI. THE INVERSION SYMMETRY AND HALL VISCOSITY

For $\tau = i$ we view the inversion symmetry as the symmetry under transformations $y_i \rightarrow -y_i$ and $x_i \rightarrow x_i, \forall i$ around x -axis or $y_i \rightarrow y_i$ and $x_i \rightarrow -x_i, \forall i$ around y -axis. For $\tau \neq i$, as we already noted, it can be generalized by adding $\tau_1 \rightarrow -\tau_1$ transformation. The symmetry has to be incorporated in the ground state wave function, more precisely in its neutral part, when we discuss the system with rectangular shape (or deformed rectangular shape, $\tau \neq i$) and our aim is the calculation of the Hall viscosity.

In the case of the Fermi-liquid-like state two constructions stand out at $\tau = i$ (and their generalizations for $\tau \neq i$) for the Fermi part

$$\mathcal{A} \left\{ \prod_{\text{over slices in k space}} \left[\prod_{i < j; i, j \in \text{slice}} \sin \left\{ \frac{\pi}{L_y} (y_i - y_j) \right\} \cos \left\{ \frac{\pi}{L_x} (x_i - x_j) \right\} \right] \right\}, \quad (a) \quad (30)$$

with the notation that we explained previously, and

$$\mathcal{A} \left\{ \prod_{\text{over slices in k space}} \left[\prod_{i < j; i, j \in \text{slice}} \cos \left\{ \frac{\pi}{L_y} (y_i - y_j) \right\} \sin \left\{ \frac{\pi}{L_x} (x_i - x_j) \right\} \right] \right\}. \quad (b) \quad (31)$$

They are explicitly invariant under the inversion symmetry transformations. The constructions are valid for both circular and rectangular Fermi surface. In Appendix B we display the functions (a) and (b) in the case of 4 particles. In that case it can be easily shown that the state - construction that is $\mathcal{A} \{ \sin \{ \frac{\pi}{L_y} (y_1 - y_2) \} \cos \{ \frac{\pi}{L_y} (y_3 - y_4) \} \dots \}$, is identical to zero. As the square of the inversion symmetry is equal to identity in general we expect that wave functions (a) and (b) represent two degenerate ground states and two independent sectors of the Fermi liquid.

Throughout the paper we discussed the case (a) for the Fermi-liquid-like state and concluded that, in the thermodynamic limit, around $\tau = i$ the Hall viscosity is equal to the one of Laughlin states and that our expectation is that for general $\tau \neq i$ this will still be true in the same limit. If we try similar arguments in the case (b) we can come to the expectation that, due to the cosine functions in the dependence on y 's, no change in the overall scaling with τ_2 will occur and for this construction the Hall viscosity is independent of τ and equal to the one of the Laughlin states.

The single particle Hamiltonian that describes the evolution of the part of the Fermi-liquid-like state that sees magnetic field is not invariant under the inversion symmetry and the “true” energetics of the problem at $\nu = 1$ (fermionic at $\nu = 1/2$) will certainly differentiate between the two possibilities for the ground state: with Fermi parts (a) and (b). We expect that the construction with Fermi part (a), irrespective whether the ground state is non-degenerate or degenerate will make a ground state as it can smoothly evolve from the thin torus limit and its Fermi part, Eq.(15), when the gauge is fixed so that the Gaussians are along y -axis. The construction with (b) Fermi part may appear as an additional sector.

VII. DISCUSSION AND CONCLUSIONS

In this paper we discussed the Berry curvature calculations of the Hall viscosity for the unprojected to the LLL wave function of the Fermi-liquid-like state. We concluded that in the linear response, with small deformation of the system and in the thermodynamic limit, the Hall viscosity takes the value characteristic for the Laughlin states (Eq.(7)). We presented arguments that the value is the same even for general deformations in the same limit.

The preprint in Ref. 14 appeared very recently when we were in the process of finishing of the present paper. There the claim is made, on the basis of the Berry curvature formula applied to the wave functions in the LLL (or projected to a definite LL), that at $\nu = 1/2$, irrespective whether the state is incompressible or not, if the Hamiltonian is particle-hole symmetric, the Hall viscosity acquires the Laughlin value. The value is the same irrespective of the deformation (τ). Though our analysis is on the unprojected (to the LLL) wave function of Fermi-liquid-like state, we agree on the value of the Hall viscosity for the state at $\nu = 1/2$ in the thermodynamic limit. For a general τ it is surprising that the same value of the Hall viscosity is maintained in the LLL^{14,15}, and somehow has to be reconciled with the expected quantization in the incompressible states. A way out is to claim that the Fermi-liquid-like state has the dissipative (symmetric) viscosity nonzero¹, but still the quantization of the Hall viscosity for the compressible state in the LLL undermines our expectation that in the Hall viscosity we have yet another characteristic of the incompressible quantum Hall states that is quantized i.e. has a constant value under small changes (perturbations) of the Hamiltonian. (In other words even gapless phases may have an invariant such as the Hall viscosity.)

On the other hand the Fermi-liquid-like state and the firmly established phase³ at $\nu = 1/2$ may be viewed as some kind of a critical state where effective particle and hole physics and two Jain's sequences of particle states (from

$\nu = 1/3$) and hole states (from $\nu = 2/3$) meet. The situation is somewhat similar or reminiscent of the graphene and the critical behavior of the neutrality point of the Dirac fermions¹⁶. Nevertheless it looks conclusive¹⁴ no critical behavior in the case of the state at $\nu = 1/2$ and the Hall viscosity is expected.

Our study shows that the Hall viscosity of the unprojected Fermi-liquid-like state at general τ may deviate from the Laughlin state value for finite number of particles. Maybe the behavior for finite number of particles can not be explained by non-interacting or weakly-interacting CF physics if we stay in the LLL as it involves higher LL physics i.e. all energy scales, which may reflect its critical nature.

In the adiabatic transport theory that we apply our basic assumption is that flux changing excitations are not relevant or higher in energy for the calculation of the Hall viscosity. The result of Ref. 14 seems to give credence to this approach. Within assumptions made, we established that the Fermi-liquid-like state in the thermodynamic limit in the linear response has the value of Hall viscosity equal to the value of Laughlin states. We hope that our analysis will help further elucidation of the problem and the search for the final answer.

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Appendix A

We will prove the formula Eq.(6) for the ground state function that is holomorphic in τ variable except for the normalization, which is the case also with the Laughlin wave function. The normalized wave function is $\Psi_0 = \frac{\Psi_L}{\sqrt{Z}}$ where Ψ_L depends on particle coordinates and τ only. We want to calculate:

$$Im \frac{\partial \langle \Psi_0 |}{\partial \tau_1} \frac{\partial | \Psi_0 \rangle}{\partial \tau_2}. \quad (A1)$$

First we have for $\tau_i; i = 1, 2$

$$\frac{\partial | \Psi_0 \rangle}{\partial \tau_i} = \frac{1}{\sqrt{Z}} \frac{\partial | \Psi_L \rangle}{\partial \tau_i} - \frac{1}{2} \frac{\partial \ln Z}{\partial \tau_i} | \Psi_0 \rangle. \quad (A2)$$

Therefore

$$Im \frac{\partial \langle \Psi_0 |}{\partial \tau_1} \frac{\partial | \Psi_0 \rangle}{\partial \tau_2} = Im \left\{ \frac{1}{Z} \frac{\partial \langle \Psi_L |}{\partial \tau_1} \frac{\partial | \Psi_L \rangle}{\partial \tau_2} - \frac{1}{2} \frac{1}{\sqrt{Z}} \frac{\partial \ln Z}{\partial \tau_1} \langle \Psi_0 | \frac{\partial | \Psi_L \rangle}{\partial \tau_2} - \frac{1}{2} \frac{1}{\sqrt{Z}} \frac{\partial \ln Z}{\partial \tau_2} \frac{\partial \langle \Psi_L |}{\partial \tau_1} | \Psi_0 \rangle \right\}. \quad (A3)$$

$| \Psi_L \rangle$ is holomorphic in τ , therefore:

$$\frac{\partial | \Psi_L \rangle}{\partial \bar{\tau}} = \frac{\partial | \Psi_L \rangle}{\partial \tau_1} + i \frac{\partial | \Psi_L \rangle}{\partial \tau_2} = 0 \quad \text{and} \quad \frac{\partial | \Psi_L \rangle}{\partial \tau} = \frac{\partial | \Psi_L \rangle}{\partial \tau_1} - i \frac{\partial | \Psi_L \rangle}{\partial \tau_2} = 0. \quad (A4)$$

Then

(1)

$$\frac{\partial \langle \Psi_L |}{\partial \tau} \frac{\partial | \Psi_L \rangle}{\partial \bar{\tau}} = \frac{\partial \langle \Psi_L |}{\partial \tau_1} \frac{\partial | \Psi_L \rangle}{\partial \tau_1} + \frac{\partial \langle \Psi_L |}{\partial \tau_2} \frac{\partial | \Psi_L \rangle}{\partial \tau_2} + i \frac{\partial \langle \Psi_L |}{\partial \tau_1} \frac{\partial | \Psi_L \rangle}{\partial \tau_2} - i \frac{\partial \langle \Psi_L |}{\partial \tau_2} \frac{\partial | \Psi_L \rangle}{\partial \tau_1} = 0. \quad (A5)$$

It follows that

$$\frac{\partial \langle \Psi_L |}{\partial \tau_1} \frac{\partial | \Psi_L \rangle}{\partial \tau_1} + \frac{\partial \langle \Psi_L |}{\partial \tau_2} \frac{\partial | \Psi_L \rangle}{\partial \tau_2} - 2 Im \frac{\partial \langle \Psi_L |}{\partial \tau_1} \frac{\partial | \Psi_L \rangle}{\partial \tau_2} = 0, \quad (A6)$$

(2)

$$\frac{\partial \ln Z}{\partial \tau_2} = \frac{1}{Z} (\langle \Psi_L | \frac{\partial | \Psi_L \rangle}{\partial \tau_2} + \frac{\partial \langle \Psi_L |}{\partial \tau_2} | \Psi_L \rangle) = -\frac{1}{\sqrt{Z}} 2 Im \langle \Psi_L | \frac{\partial | \Psi_L \rangle}{\partial \tau_1}, \quad (A7)$$

(3)

$$\frac{\partial \ln Z}{\partial \tau_1} = \frac{1}{Z} (\langle \Psi_L | \frac{\partial |\Psi_L\rangle}{\partial \tau_1} + \frac{\partial \langle \Psi_L|}{\partial \tau_1} | \Psi_L \rangle) = \frac{1}{\sqrt{Z}} 2 \operatorname{Im} \langle \Psi_L | \frac{\partial |\Psi_L\rangle}{\partial \tau_2}. \quad (\text{A8})$$

From (A7) and (A8) it follows

$$\operatorname{Im} \frac{\partial \langle \Psi_0 |}{\partial \tau_1} \frac{\partial |\Psi_0\rangle}{\partial \tau_2} = \operatorname{Im} \left\{ \frac{1}{Z} \frac{\partial \langle \Psi_L |}{\partial \tau_1} \frac{\partial |\Psi_L\rangle}{\partial \tau_2} \right\} - \frac{1}{4} \left(\frac{\partial \ln Z}{\partial \tau_1} \right)^2 - \frac{1}{4} \left(\frac{\partial \ln Z}{\partial \tau_2} \right)^2. \quad (\text{A9})$$

Because

$$\frac{\partial^2 \ln Z}{\partial \tau_i^2} = \frac{\partial}{\partial \tau_i} \frac{1}{Z} \frac{\partial Z}{\partial \tau_i} = -\frac{1}{Z^2} \left(\frac{\partial Z}{\partial \tau_i} \right)^2 + \frac{1}{Z} \frac{\partial^2 Z}{\partial \tau_i^2}, \quad (\text{A10})$$

and because of (A6)

$$\Delta Z = \frac{\partial^2}{\partial \tau \partial \bar{\tau}} Z = \frac{\partial \langle \Psi_L |}{\partial \bar{\tau}} \frac{\partial |\Psi_L\rangle}{\partial \tau} = \frac{\partial \langle \Psi_L |}{\partial \tau_1} \frac{\partial |\Psi_L\rangle}{\partial \tau_1} + \frac{\partial \langle \Psi_L |}{\partial \tau_2} \frac{\partial |\Psi_L\rangle}{\partial \tau_2} + 2 \operatorname{Im} \frac{\partial \langle \Psi_L |}{\partial \tau_1} \frac{\partial |\Psi_L\rangle}{\partial \tau_2} = 4 \operatorname{Im} \frac{\partial \langle \Psi_L |}{\partial \tau_1} \frac{\partial |\Psi_L\rangle}{\partial \tau_2}, \quad (\text{A11})$$

we have

$$\operatorname{Im} \frac{\partial \langle \Psi_0 |}{\partial \tau_1} \frac{\partial |\Psi_0\rangle}{\partial \tau_2} = \frac{1}{4} \Delta \ln Z. \quad (\text{A12})$$

Appendix B

For four particles the construction in Eq.(18) or Eq.(30) is

$$\begin{aligned} \Psi_b = & \sin\left\{\frac{\pi}{L_y}(y_1 - y_2)\right\} \sin\left\{\frac{\pi}{L_y}(y_3 - y_4)\right\} \cos\left\{\frac{\pi}{L_x}(x_1 - x_2)\right\} \cos\left\{\frac{\pi}{L_x}(x_3 - x_4)\right\} \\ & - \sin\left\{\frac{\pi}{L_y}(y_1 - y_3)\right\} \sin\left\{\frac{\pi}{L_y}(y_2 - y_4)\right\} \cos\left\{\frac{\pi}{L_x}(x_1 - x_3)\right\} \cos\left\{\frac{\pi}{L_x}(x_2 - x_4)\right\} \\ & + \sin\left\{\frac{\pi}{L_y}(y_1 - y_4)\right\} \sin\left\{\frac{\pi}{L_y}(y_2 - y_3)\right\} \cos\left\{\frac{\pi}{L_x}(x_1 - x_4)\right\} \cos\left\{\frac{\pi}{L_x}(x_2 - x_3)\right\}, \end{aligned} \quad (\text{B1})$$

and the one in Eq.(31) is

$$\begin{aligned} \Psi_a = & \cos\left\{\frac{\pi}{L_y}(y_1 - y_2)\right\} \cos\left\{\frac{\pi}{L_y}(y_3 - y_4)\right\} \sin\left\{\frac{\pi}{L_x}(x_1 - x_2)\right\} \sin\left\{\frac{\pi}{L_x}(x_3 - x_4)\right\} \\ & - \cos\left\{\frac{\pi}{L_y}(y_1 - y_3)\right\} \cos\left\{\frac{\pi}{L_y}(y_2 - y_4)\right\} \sin\left\{\frac{\pi}{L_x}(x_1 - x_3)\right\} \sin\left\{\frac{\pi}{L_x}(x_2 - x_4)\right\} \\ & + \cos\left\{\frac{\pi}{L_y}(y_1 - y_4)\right\} \cos\left\{\frac{\pi}{L_y}(y_2 - y_3)\right\} \sin\left\{\frac{\pi}{L_x}(x_1 - x_4)\right\} \sin\left\{\frac{\pi}{L_x}(x_2 - x_3)\right\}. \end{aligned} \quad (\text{B2})$$

i.e. with x 's and y 's interchanged.

The wave functions Ψ_a and Ψ_b can be represented by their configurations in k -space. Below each configuration describes the placements of four fermions in the corners of a square that correspond to allowed values of four momenta.

$$\begin{aligned} \Psi_{a(b)} = & + \frac{3|1}{2|4} \pm \frac{1|3}{4|2} \pm \frac{2|4}{3|1} + \frac{4|2}{1|3} \\ & \mp \frac{4|1}{2|3} - \frac{1|4}{3|2} - \frac{2|3}{4|1} \mp \frac{3|2}{1|4} \\ & - \frac{2|1}{3|4} \mp \frac{1|2}{4|3} \mp \frac{3|4}{2|1} - \frac{4|3}{1|2} \\ & \pm \frac{4|1}{3|2} + \frac{1|4}{2|3} + \frac{3|2}{4|1} \pm \frac{2|3}{1|4} \\ & + \frac{2|1}{4|3} \pm \frac{1|2}{3|4} \pm \frac{4|3}{2|1} + \frac{3|4}{1|2} \\ & \mp \frac{3|1}{4|2} - \frac{1|3}{2|4} - \frac{4|2}{3|1} \mp \frac{2|4}{1|3}. \end{aligned} \quad (\text{B3})$$

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